Levels of Learning

In teaching algebraic thinking it is important that students reach the abstract level of learning. According to Miller and Mercer (1997, 1993), long-term functioning in abstract algebraic thinking is achieved by starting students in the concrete level and progressing through the representational level. Students should develop metacognitive learning strategies, to enable them to problem solve on an abstract level. This ASK model was designed for K-12 students and focuses on the Levels of Learning: Concrete-Representational-Abstract (CRA) (Miller & Mercer 1997; Witzel, Smith & Brownell, 2001)

The following are definitions of each of the CRA levels of learning:

- **Concrete Level**- A teaching method that uses actual objects in teaching mathematical concepts.

- **Representational Level**- A teaching method that uses pictures, tally marks, diagrams, and drawings in teaching mathematical concepts. These pictorial representations relate directly to the manipulatives and provide scaffolding that enables students to solve numeric problems at the abstract level.

- **Abstract Level**- A teaching method that uses written words (including Braille), symbols (such as variables or numerals), or verbal expressions (including sign language) in teaching mathematical concepts.

The following table provides a graphic organizer on how CRA, used effectively, maximizes instruction in algebraic thinking. To be effective, CRA should be used in many ways to help students learn algebra. Primary teachers should begin by teaching using concrete manipulatives and model the use of manipulatives to start developing an algebraic thinking concept. Conversely, a secondary teacher could initially teach at the abstract level and use representational examples to illustrate the algebraic thinking concept. Teachers should use CRA as a method

of analyzing student learning and understanding to help scaffold instruction of algebraic thinking concepts at all grade levels.

Illustration of CRA use in teaching algebraic thinking

Teachers

Students	C	R	A
c	Cc	Rc	Ac
r	Cr	Rr	Ar
a	Ca	Ra	Aa

Key

C=Concrete

R=Representational

A=Abstract

Cc=Teacher teaches concretely while the students practice concretely.

For example: Teacher uses counting cubes to teach the concept of multiplication while the students have their own counting cubes to practice the new concept.

Rc=Teacher uses representation while students practice concretely.

For example: Teacher teaches the concept of multiplication on the whiteboard while the students use counting cubes to practice the concept.

Ar= teacher teaches abstractly while students practice representationally.

For example: Teacher teaches the concept of multiplication through word problems while the students practice the concept using tally marks on the paper.

The following pages give a more detailed explanation of CRA as well as more examples, K-12.

Algebraic Thinking Strategies and Tools for Grades K-12 Using CRA

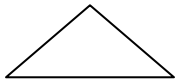
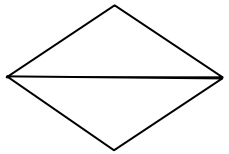
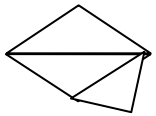
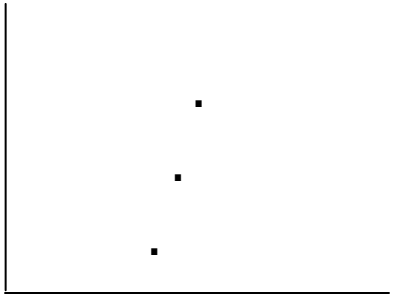
The following table illustrates how algebraic thinking strategies and tools may be used in CRA instruction at each grade level. It is important to note that items under each listing could help teach all or some of the levels depending on the context in which is used. This information is useful for elementary grades, as it demonstrates the importance of students mastering algebraic thinking concepts later taught. Secondary teachers will build on this prior knowledge and expand student learning to more abstract concepts.

CRA Strategies and Tools					
Concrete	**Grades**	**Representational**	**Grades**	**Abstract**	**Grades**
▣ Abacus	4-12	▣ Diagrams	K-12	▣ Oral Conversation	K-12
▣ Counting Rods/Blocks	K-5	▣ Blue Print	3-12	▣ Story Telling	K-12
▣ Unifix Cubes	K-5	▣ Ruler, straight edge	K-12	▣ Understand mathematical and real-world processes	K-12
▣ Counters	K-12	▣ Compass	3-12		
▣ Dice	K-12	▣ Calculators	3-12		
▣ Coins	2-12	▣ Graphing Calculators	6-12		
▣ Base Ten Block	K-12	▣ Computer Graphing Software	6-12	▣ Independent Problem Solving	3-12
▣ Edibles: M&M's & Starburst	3-12	▣ Pictures of manipulatives	K-12	▣ Essays	3-12
▣ Algebra Tiles	5-12	▣ Venn Diagrams	3-12	▣ Spreadsheet Software	3-12
		▣ Maps	3-12	▣ Tables	3-12
		▣ Drawings	K-12		
		▣ Picture Graphs	3-12		

The table below addresses Algebra Strand 1, *Understanding patterns, relations, and functions (NCTM 2000),* illustrates an example for levels K-2, 3-5, 6-8, and 9-12 and shows how the concept is developed through the curriculum.

Algebra Strand 1 Understanding Patterns, Relations and Functions		
Grades K-2 Classify/Sort/Order Objects		
Problem: Student experiment with patterns and determine the next item or items in a pattern		
C	R	A
Children group themselves by gender, by size, etc. Use other objects (dogs, animals, textbooks, toys, books)	Decorate paper doll worksheet to represent themselves, then sort by attributes. Use pictures to classify and determine patterns	Explain verbally the pattern and also explain how to determine the next item in the pattern

Algebra Strand 1 Understanding Patterns, Relations and Functions		
Grades 3-5 Understand the pattern and predict the next number in the numeric pattern, Example 1,4,9,16,25…		
Problem: Student experiment with patterns and determine the next item or items in a pattern		
C	R	A
Create patterns with manipulatives. Ex: Multi-colored or base 10 pattern blocks, Tiles	Pictorial pattern of blocks.	Verbal or written description of pattern

Algebra Strand 1 Understanding Patterns, Relations and Functions		
Grades 6-8 Determine the symbolic rule of a pattern		
Problem: Find the number of total sides of connecting equilateral triangles.		
C	R	A
Use toothpicks to analyze a set of patterns. Example t s 1  2  3 	Use input/output tables and line graphs to investigate the rule. t | s 1 | 3 2 | 5 3 | 7 4 | 9 r | ? t = # of triangles s = # of toothpicks or line segment 	Write a symbolic rule. Explain the continuing relation. Describe how the pattern is linear. 2t + 1 = s

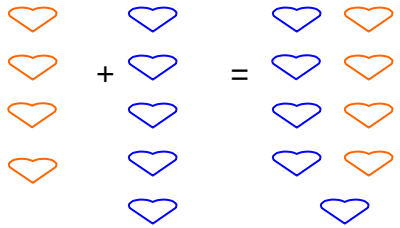
<table>
<tr><td colspan="3">Algebra Strand 1 Understanding Patterns, Relations and Functions</td></tr>
<tr><td colspan="3">Grades 9-12 Use position number to predict a number in a pattern</td></tr>
<tr><td colspan="3">Problem: Find the number of total sides of connecting equilateral triangles.</td></tr>
<tr><td>C</td><td>R</td><td>A</td></tr>
<tr><td>Use toothpicks to create a set of connected triangular patterns.</td><td>On dot paper, draw the patterns made with triangles, and graph them.
Graph
Triangles x Axis

Total # sides y Axis</td><td>Determine the formula for the total number of sides as indicated.
<table><tr><td># Triangles</td><td># Sides</td></tr><tr><td>1</td><td>3</td></tr><tr><td>2</td><td>5</td></tr><tr><td>3</td><td>7</td></tr><tr><td>4</td><td>9</td></tr><tr><td>t</td><td>s</td></tr></table>
Show how the number of triangles relates to the number of sides using triangles as position number.
<table><tr><td>t</td><td>s</td><td>s1</td><td>s2</td><td>s3</td><td>s</td></tr><tr><td>1</td><td>3</td><td>1 + 2</td><td>t + 2</td><td>t + (t + 1)</td><td>2t + 1</td></tr><tr><td>2</td><td>5</td><td>2 + 3</td><td>t + 3</td><td>t + (t + 1)</td><td>2t + 1</td></tr><tr><td>3</td><td>7</td><td>3 + 4</td><td>t + 4</td><td>t + (t + 1)</td><td>2t + 1</td></tr><tr><td>4</td><td>9</td><td>4 + 5</td><td>t + 5</td><td>t + (t + 1)</td><td>2t + 1</td></tr></table></td></tr>
</table>

The following table addresses Algebra Strand 2, *Represent and analyze mathematical situation and structures using algebraic symbols* (NCTM, 2000), illustrates an example for levels K-2, 3-5, 6-8, and 9-12 and shows how the concept is developed through the curriculum.

<table>
<tr><td colspan="3">Algebra Strand 2: Represent and analyze mathematical situations and structures using algebraic symbols</td></tr>
<tr><td colspan="3">Grades K-2 Illustrate general principles and properties, such as the Commutative Property, using specific numbers</td></tr>
<tr><td colspan="3">Problem: Students identify concrete or representational patterns and then write equations of the pattern.</td></tr>
<tr><td>C</td><td>R</td><td>A</td></tr>
<tr><td>(Pre-concrete-setting relevance) Read the book: Twelve Ways to Get to Eleven. Have children act out combining different addends (Sums to 11). Manipulate beans (painted sides) or two sided counters to combine different addends. Example, “How many red and white beans make 11?”

Play “Shake, Rattle, and Roll.” Shake counters, (containing 11 beans each painted red on one side and white on the other), and see different combinations.</td><td>Pictorial representation of the pattern. Using colors to illustrate the pattern. Use cut up representations and paste

Make pictures of all the different combinations
 
</td><td>Students justify their thought process by writing equations in math journal.

Student write an equation like:
4 +5 = 9
6 + 3 = 11</td></tr>
</table>

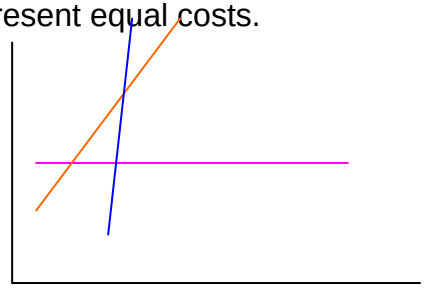
Algebra Strand 2: Represent and analyze mathematical situations and structures using algebraic symbols		
Grades 3-5 Represent the idea of a variable as an unknown quantity using a letter or a symbol		
Problem: Which milk is the least expensive per gallon: quart of milk, $1.00, ½ gallon of milk, $1.79, and a gallon of milk, $3.98.		
C	R	A
Have various size milk jugs available. Use the milk jugs to compare equivalents. Conversion Chart 1 Gallon = 4 quarts ½ Gallon = 2 quarts 1 Gallon – 2 ½ gallons	Draw a scale picture of a quart of milk, $1.00, a picture of a ½ gallon of milk, $1.79, and a picture of a gallon of milk, $3.98. Draw a possible solution to this problem	Verbally explain the answer. 2 quarts of milk= $2.00 and ½ gallon of milk =$1.79 therefore the ½ gallon is a better price. 2 ,½ gallons = $3.58 and 1 gallon = $3.98, therefore the ½ gallons are a better price.

<table>
<tr><td colspan="3">Algebra Strand 2: Represent and analyze mathematical situations and structures using algebraic symbols</td></tr>
<tr><td colspan="3">Grades 6-8 Recognize and generate equivalent forms for simple algebraic expressions and solve linear equations</td></tr>
<tr><td colspan="3">Problem: Which video store offers the best deal when renting less than seven videos? Videoworld: $30 membership fee and cost of video, $2.50. Videorama: $20 membership fee and cost of video, $4.00. Which video store offers the best deal when renting seven or more videos? When will the cost be equal?</td></tr>
<tr><td>C</td><td>R</td><td>A</td></tr>
<tr>
<td>Use play money and a shopping list (bills and coins) to solve the problem.
Use base ten blocks to concretely show pattern clearly.
Concrete objects should lead to the end result … patterns within linear functions</td>
<td>Use tables and charts to illustrate
Blockbuster Video Superstore
<table>
<tr><td># of Videos</td><td>Total Cost (v)</td><td>Total Cost (t)</td></tr>
<tr><td>1</td><td>32.50</td><td>24.00</td></tr>
<tr><td>2</td><td>35.00</td><td>28.00</td></tr>
<tr><td>3</td><td>37.50</td><td>32.00</td></tr>
<tr><td>4</td><td>40.00</td><td>36.00</td></tr>
<tr><td>5</td><td>42.50</td><td>40.00</td></tr>
<tr><td>6</td><td>45.00</td><td>44.00</td></tr>
<tr><td>7</td><td>47.50</td><td>48.00</td></tr>
</table>
</td>
<td>Students write equations and solve the problem.
Videoworld: 30 + 2.50 v = t
Videorama: 20 + 4 v = t</td>
</tr>
</table>

Algebra Strand 2: Represent and analyze mathematical situations and structures using algebraic symbols

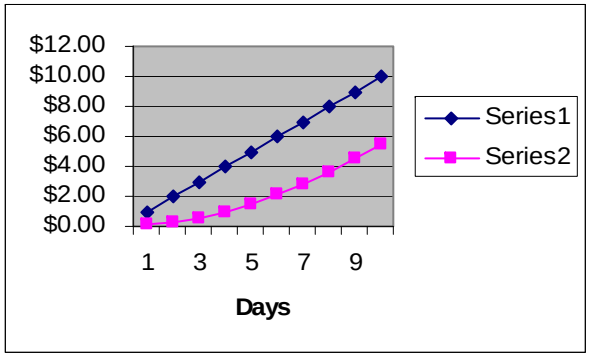
Grades 9-12 Given information, the student will write a system of equations, use a graphing calculator and or graph paper to model the situation and make a decision based on the graph

Problem: Penny, Mary Ann, and Lynn are going to order Chinese take-out food. There are 3 local restaurants of equal quality; and they are located within a block of one another, but the delivery charges vary. How do the girls determine which restaurant to order from?

C	R	A
Students will use a map to find locations, and measure with a ruler to determine distances. Note: The map is representational. Measuring on the map is concrete.	Represent problem with a table to solve. Linh Vu's: $1.50 plus $0.50 per half mile Lucy Lu's: $3.50 plus $0.25 per half mile Ming Garden's: $7.00 flat fee Graph the lines to represent the solution. On the graph, the intersections represent equal costs. 	Write the linear equations to represent the cost as a function of distance. LV = 1.50 + .5x LL = 3.50 + .25x MG = 7 Using the graph, decide which restaurant is the best deal, depending on distance. Girls would buy from: Linh Vu's if they lived 0-8 miles away Lucy Lu's if they lived from 8-14 miles away Ming Garden if they lived 14 or more miles away

The following table addresses Algebra Strand 3, *Use mathematical models to represent and understand quantitative relationships* (NCTM, 2000), illustrates an example for levels K-2, 3-5, 6-8, and 9-12 and shows how the concept is developed through the curriculum.

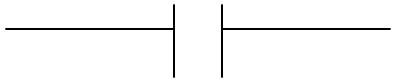
<table>
<tr><th colspan="3">Algebra Strand 3: Use mathematical models to represent and understand quantitative relationships</th></tr>
<tr><td colspan="3">Grades K-2: Model situations that involve addition and subtraction of whole numbers using objects, pictures, and symbols</td></tr>
<tr><td colspan="3">Problem: John’s parents have agreed to pay him an allowance for the next 10 days. They will pay him in one of two ways. Help John decide which of the options he should take. Option 1: $.50 a day for 10 days or Option 2: $0.10 the first day, $0.20 the second day, $0.30 the third day and so on until the 30th day</td></tr>
<tr><td>C</td><td>R</td><td>A</td></tr>
<tr><td>Students manipulate play money for first 5 days for both options.</td><td>Fill in T chart (First 5 days).
<table><tr><th>Days</th><th>Option 1</th><th>Option 2</th></tr><tr><td>1</td><td>$0.50</td><td>$0.10</td></tr><tr><td>2</td><td>$1.00</td><td>$0.30</td></tr><tr><td>3</td><td>$1.50</td><td>$0.60</td></tr><tr><td>4</td><td>$2.00</td><td>$1.00</td></tr><tr><td>5</td><td>$2.50</td><td>$1.50</td></tr></table></td><td>With the class, graph the 10 days of both options.
After this compare the graphs as a class.</td></tr>
</table>

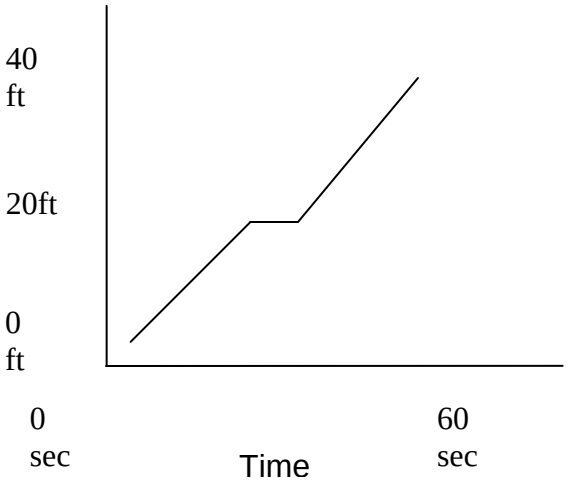
Algebra Strand 3: Use mathematical models to represent and understand quantitative relationships		
Grades 3-5: Model problem situations with objects and use representations such as graphs, tables, and equations to draw conclusions		
Problem: John's parents have agreed to pay him an allowance for the next 30 days. They will pay him in one of two ways. Help John decide which of the options he should choose. • Option 1: $1.00 a day for 30 days or • Option 2: $0.10 the first day, $0.20 the second day, $0.30 the third day and so on until the 30th day.		
C	R	A
Students manipulate play money for 10 days for both options.	Fill in T chart for the first 10 days. Days Option 1 Option 2 1 $1.00 $0.10 2 $2.00 $0.30 3 $3.00 $0.60 4 $4.00 $1.00 5 $5.00 $1.50 6 $6.00 $2.10 7 $7.00 $2.80 8 $8.00 $3.60 9 $9.00 $4.50 10 $10.00 $5.50	With the class, graph the first 10 days of each option. Discuss appearance of graph. Make prediction of what the graph will look like after 30 days. Have students complete the worksheet and graph for all 30 days. After this do "Comparing" worksheet. 

Algebra Strand 3: Use mathematical models to represent and understand quantitative relationships		
Grades 6-8: Use mathematical models to represent and understand quantitative relationships		
Problem: Grandma gave Joe $100.00. Bank X offered him 1% interest with no monthly fee while Bank Y offered 5% interest with a monthly fee of $10.00. Should Joe open his savings account at Bank X or Bank Y?		
C	R	A
Use play money to add money equivalent to the interest to the original 100 dollars.	Use input/output table to construct one visual to show the difference in growth. Given specific information about savings accounts, create a table and graph to represent growth. Month / Bank X 1% / Bank Z 5% + $10 1 — 101 — 95 2 — 102.01 — 89.75 3 — 103.03 — 84.24 4 — 104.06 — 78.45	Write an equation for each bank to predict monthly growth. Show on the graph the prediction for future growth over time. b + .01b = n b + .05b - $10 = n (b is balance, n is new balance)

<table>
<tr><td colspan="3">Algebra Strand 3: Use mathematical models to represent and understand quantitative relationships</td></tr>
<tr><td colspan="3">Grades 9-12: Use mathematical models to represent and understand quantitative relationships</td></tr>
<tr><td colspan="3">Problem: Margie has $1000 to invest. She plans to put it into a savings account. She must determine the bank that will pay the highest amount of interest on her investment, and then calculate the length of time it will take for her bank balance to increase to $2000. Is this investment plan practical? What other investment methods might Margie consider?</td></tr>
<tr><td>C</td><td>R</td><td>A</td></tr>
<tr><td>Students will use play money to represent and investigate various saving account rates.</td><td>Students will utilize newspapers and online sources to investigate various savings account rates. They will select the bank that pays the highest rate of interest. (Students should use graphs.)
Bank of America rate (R) = 4%
Investment amount (P) = $1000 (principle)
Interest amount (I) = ___?___
Number of years (T) = ____?___</td><td>Interest rate for each year:
$1000 x .04 x 1 = $40
p x r x t = I
Interest earned over time:
$2000-$1000=$1000
Interest (i) + Principle (p) = Total Amount (TA)
Find TA
<table>
<tr><th>Yr</th><th>Interest
I= p x r x t</th><th>+</th><th>Principle
R</th><th>Total
Amt TA</th></tr>
<tr><td>1</td><td>1000x.04x1=40</td><td>+</td><td>$1000</td><td>$1040</td></tr>
<tr><td>2</td><td>1000 x .04 x2 = 80</td><td>+</td><td>$1000</td><td>$1080</td></tr>
<tr><td>3</td><td>1000 x .04 x3 = 120</td><td>+</td><td>$1000</td><td>$1120</td></tr>
<tr><td>T</td><td>1000 x .04 x T = 1000</td><td>+</td><td>$1000</td><td>$2000</td></tr>
</table>
1000 x .04 x t = 1000
40t = 1000
t = 25
It will take 25 years for Margie to have a total amount of $2000.</td></tr>
</table>

The following table addresses Algebra Strand 4, *Analyze change in various contexts* (NCTM, 2000), illustrates an example for levels K-2, 3-5, 6-8, and 9-12 and shows how the concept is developed through the curriculum.

<table>
<tr><th colspan="3">Algebra Strand 4: Analyze change in various contexts</th></tr>
<tr><th colspan="3">Grades K-2: Describe qualitative and quantitative change (time, distance, and rates)</th></tr>
<tr><td colspan="3">Problem: Sara needs to walk 40 feet. Half way she needs to tie her shoes. Then she continues her walk. How long did it take her to travel to the path?</td></tr>
<tr><td>C</td><td>R</td><td>A</td></tr>
<tr><td>Measure the 40 ft. to establish the path. Using a timer, have the students travel the path stopping once to tie their shoes (20 ft.). Find total time elapsed from start to finish. List results on board.
Note: Main objective is for the students to understand that time continues even when they stop to tie their shoes.</td><td>Draw a picture representing previous activity. Make sure they label the time it took to complete the path.
Note: The timer is a representational tool.
Walk (time) Stop Walk(time)
  </td><td>Have students share what they experienced in this activity.
Note: Main objective is to discuss time differences and why?</td></tr>
</table>

Algebra Strand 4: Analyze change in various contexts		
Grades3-5: Concept: Investigate how a change in one variable relates to a change in a second variable		
For example: Sara needs to walk 40 feet. Half way she stops. Then she continues her walk. How long did it take her to travel to the path?		
C	R	A
Student's role-play walking with timer and floor tiles (12" square) or a pre-measured area.	Draw a linear graph and discuss its appearance based on the role-playing.  	Write a story interpreting the graph. The first segment show Sara distance increases as time increases starting at 0 seconds and 0 feet because she was not moving to 20 feet. When the line is flat, time continues to pass but Sara does not increase move any distance. Finally, after she ties her shoe, distance goes up as she continues to move with time.

Algebra Strand 4: Analyze change in various contexts		
Grades 6-8: Concept: Use graphs to analyze the nature of changes in quantities in linear relationships		
Problem: Suzie leaves her home at 7:15 am and travels on a skateboard at a rate of 4 mph. Mike, he brother leaves the same home at 7:30 am and travels on a bicycle at a rate of 7 mph. Will Mike be able to catch up with Suzie? If so, how much time will it take him to catch up with Suzie?		
C	R	A
Student's role play the math problem by walking with a timer and at the indicated rates.	Draw a line graph showing change over time. 7:30 am ——— 7 mph ———▶ 7:15 am ——— 4 mph ———▶ Arrows should be equal length because their distances will be the equal when they meet.	Distance = Rate • Time (in mph) Suzie left earlier so she traveled longer time traveled=t 15 minutes to mph 15/60 equal ¼ hour Suzie Distance d = 4(t+1/4) Mike Distance d = 7t Since their distances will be equal when they meet set the equations equal to each other 4(t+1/4) = 7t 4t + 1 = 7t 1 = 3t 1/3 = t In 1/3 of an hour or 20 minutes they will meet.

Algebra Strand 4: Analyze change in various contexts		
Grades 9-12: Concept: Approximate and interpret rates of change from graphical and numerical data		
Problem: Including the time Josephine spent tying her shoe, what was the average rate of speed to walk 400 feet?		
C	R	A
Walk and use timer, ruler, and/or Computer based lab or calculator based lab (CBL) motion detector.	Use graphing calculator to graph.	Write rate equations. $25x + 5x + 25x = 400$ $55x = 400$ $x \approx \underline{7.27}$ $r \approx .73$ ft./sec. $d=rt$